

# Defective 2-colorings of planar graphs without 4-cycles and 5-cycles

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## Abstract

Let  $G$  be a graph without 4-cycles and 5-cycles. We show that the problem to determine whether  $G$  is  $(0, k)$ -colorable is NP-complete for each positive integer  $k$ . Moreover, we construct non- $(1, k)$ -colorable planar graphs without 4-cycles and 5-cycles for each positive integer  $k$ . Finally, we prove that  $G$  is  $(d_1, d_2)$ -colorable where  $(d_1, d_2) = (4, 4), (3, 5)$ , and  $(2, 9)$ .

## 1 Introduction

Let  $G$  be a graph with the vertex set  $V(G)$  and the edge set  $E(G)$ . A  $k$ -vertex, a  $k^+$ -vertex, and  $k^-$ -vertex are a vertex of degree  $k$ , at least  $k$ , and at most  $k$ , respectively. The similar notation is applied for faces. A  $(d_1, d_2, \dots, d_k)$ -face  $f$  is a face of degree  $k$  where all vertices on  $f$  have degree  $d_1, d_2, \dots, d_k$ . If  $v$  is not on a 3-face  $f$  but  $v$  is adjacent to some 3-vertex on  $f$ , then we call  $f$  a *pendant face* of a vertex  $v$  and  $v$  is a *pendant neighbor* of a 3-vertex  $v$ . A 3-face (respectively, 2-vertex) incident to a 2-vertex (respectively, 3-face) is called a *bad 3-face* (respectively, *bad 2-vertex*). Otherwise, it is a *good 3-face* (respectively, *good 2-vertex*).

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A  $k$ -coloring  $c$  (not necessary proper) is a function  $c : V(G) \rightarrow \{1, \dots, k\}$ . Define  $V_i := \{v \in V(G) : c(v) = i\}$ . We call  $c$  a  $(d_1, d_2, \dots, d_k)$ -coloring if  $V_i$  is an empty set or the induced subgraph  $G[V_i]$  has the maximum degree at most  $d_i$  for each  $i \in \{1, \dots, k\}$ . A graph  $G$  is called  $(d_1, d_2, \dots, d_k)$ -colorable if  $G$  admits a  $(d_1, d_2, \dots, d_k)$ -coloring. Thus the four color theorem [2],[3] can be restated as every planar graphs is  $(0, 0, 0, 0)$ -colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is  $(2, 2, 2)$ -colorable [10]. Eaton and Hull [11] proved that  $(2, 2, 2)$ -colorability is optimal by showing non- $(k, k, 1)$ -colorable planar graphs for each  $k$ .

Grötzsch [12] showed that every planar graph without 3-cycles is  $(0, 0, 0)$ -colorable. The famous Steinberg's conjecture proposes that every planar graph without 4-cycles and 5-cycles is also  $(0, 0, 0)$ -colorable. Recently, this conjecture is disproved by Cohen-Addad et al [1]. One way to relax the conjecture is allowing some color classes to be improper. For every planar graph  $G$  without 4-cycles and 5-cycles, Xu, Miao, and Wang [17] proved that  $G$  is  $(1, 1, 0)$ -colorable, and Chen et al. [8] proved that  $G$  is  $(2, 0, 0)$ -colorable.

Many papers investigate  $(d_1, d_2)$ -coloring of planar graphs in various settings. Montassier and Ochem [14] constructed planar graphs of girth 4 that are not  $(i, j)$ -colorable for each  $i, j$ . Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed planar graphs of girth 6 that are not  $(0, k)$ -colorable for each  $k$ . On the other hand, for every planar graph  $G$  of girth 5, Havet and Seren [13] showed that  $G$  is  $(2, 6)$ -colorable and  $(4, 4)$ -colorable, and Choi and Raspaud [9] showed that  $G$  is  $(3, 5)$ -colorable.

Let  $G$  be a graph with the vertex set  $V(G)$  and the edge set  $E(G)$ . A graph  $G$  is called  $(d_1, d_2, \dots, d_k)$ -colorable if  $V(G)$  can be partitioned into sets  $V_1, V_2, \dots, V_k$  such that the induced subgraph  $G[V_i]$  for  $i \in [k]$  has the maximum degree at most  $d_i$ . Thus the four color theorem [2],[3] can be restated as every planar graphs is  $(0, 0, 0, 0)$ -colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is  $(2, 2, 2)$ -colorable [10]. Eaton and Hull [11] and Škrekovski [15] prove that  $(2, 2, 2)$ -colorability is optimal by showing non- $(k, k, 1)$ -colorable planar graphs for each  $k$ .

Grötzsch [12] showed that every planar graph without 3-cycles is  $(0, 0, 0)$ -colorable. The famous Steinberg's conjecture proposes that every planar graph without 4-cycles and 5-cycles is also  $(0, 0, 0)$ -colorable. Recently, this conjecture is disproved by Cohen-Addad et al [1]. One way to relax the conjecture is allowing some color classes to be improper. For every planar graph  $G$  without 4-cycles and 5-cycles, Xu, Miao, and Wang [17] proved that  $G$  is  $(1, 1, 0)$ -colorable, and Chen et al. [8] proved that  $G$  is  $(2, 0, 0)$ -colorable.

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planar graph  $G$  of girth 5, Havet and Seren [13] showed that  $G$  is  $(2, 6)$ -colorable and  $(4, 4)$ -colorable, and Choi and Raspaud [9] showed that  $G$  is  $(3, 5)$ -colorable. Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed planar graphs of girth 6 that are not  $(0, k)$ -colorable for each  $k$ . Montassier and Ochem [14] constructed planar graphs of girth 4 that are not  $(i, j)$ -colorable for any  $i, j$ .

There are many papers [4, 6, 13, 7, 5, 14] that investigate  $(d_1, d_2)$ -colorability for graphs with girth length of  $g$  for  $g \geq 6$ ; see [14] for the rich history. For example, Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed a graph in  $g_6$  (and thus also in  $g_5$ ) that is not  $(0, k)$ -colorable for any  $k$ . The question of determining if there exists a finite  $k$  where all graphs in  $g_5$  are  $(1, k)$ -colorable is not yet known and was explicitly asked in [14]. On the other hand, Borodin and Kostochka [6] and Havet and Sereni [13], respectively, proved results that imply graphs in  $g_5$  are  $(2, 6)$ -colorable and  $(4, 4)$ -colorable.

Let  $G$  be a graph without 4-cycles and 5-cycles. We show that the problem to determine whether  $G$  is  $(0, k)$ -colorable is NP-complete for each positive integer  $k$ . Moreover, we construct non- $(1, k)$ -colorable planar graphs without 4-cycles and 5-cycles for each positive integer  $k$ . Finally, we prove that  $G$  is  $(d_1, d_2)$ -colorable where  $(d_1, d_2) = (4, 4), (3, 5)$ , and  $(2, 9)$ .

## 2 NP-completeness of $(0, k)$ -colorings

**Theorem 1.** [14] *Let  $g_{k,j}$  be the largest integer  $g$  such that there exists a planar graph of girth  $g$  that is not  $(k, j)$ -colorable. The problem to determine whether a planar graph with girth  $g_{k,j}$  is  $(k, j)$ -colorable for  $(k, j) \neq (0, 0)$  is NP-complete.*

**Theorem 2.** *The problem to determine whether a planar graph without 4-cycles and 5-cycles is  $(0, k)$ -colorable is NP-complete for each positive integer  $k$ .*

**Proof.** We use a reduction from the problem in Theorem 1 to prove that  $(0, k)$ -coloring for planar graph without 4-cycles and 5-cycles. From [14],  $6 \leq g_{0,1} \leq 10$ . Let  $G$  be a graph of girth  $g_{0,1}$ . Take  $k - 1$  copies of 3-cycles  $v_i v'_i v''_i$  ( $i = 1, \dots, k - 1$ ) for each vertex  $v$  of  $G$ . The graph  $H_k$  is obtained from  $G$  by identifying  $v_i$  (in a 3-cycle  $v_i v'_i v''_i$ ) to  $v$  for each vertex  $v$ . The resulting graph  $H_k$  has neither 4-cycles nor 5-cycles.

Suppose  $G$  has a  $(0, 1)$ -coloring  $c$ . We extend a coloring to  $c(v'_i) = 1$  and  $c(v''_i) = 2$  for each vertex  $v$  and each  $i = 1, \dots, k - 1$ . One can see that  $c$  is a  $(0, k)$ -coloring of  $H_k$ . Suppose  $H_k$  has a  $(0, k)$ -coloring  $c$ . Consider  $v \in V(G)$  with  $c(v) = 2$ . By construction,  $v$  has at least

$k - 1$  neighbors with the same color in  $V(H_k) - V(G)$ . Thus  $v$  has at most one neighbor with the same color in  $V(H_k) - V(G)$ . It follows that  $c$  with restriction to  $V(G)$  is a  $(0, 1)$ -coloring of  $G$ . Hence  $G$  is  $(0, 1)$ -colorable if and only if  $H_k$  is  $(0, k)$ -colorable. This completes the proof.  $\square$

### 3 Non- $(1, k)$ -colorable planar graphs without 4-cycles and 5-cycles

We construct a non- $(1, k)$ -colorable planar graph  $G$  without 4-cycles and 5-cycles. Consider the graph  $H_{u,v}$  shown in Figure 1.

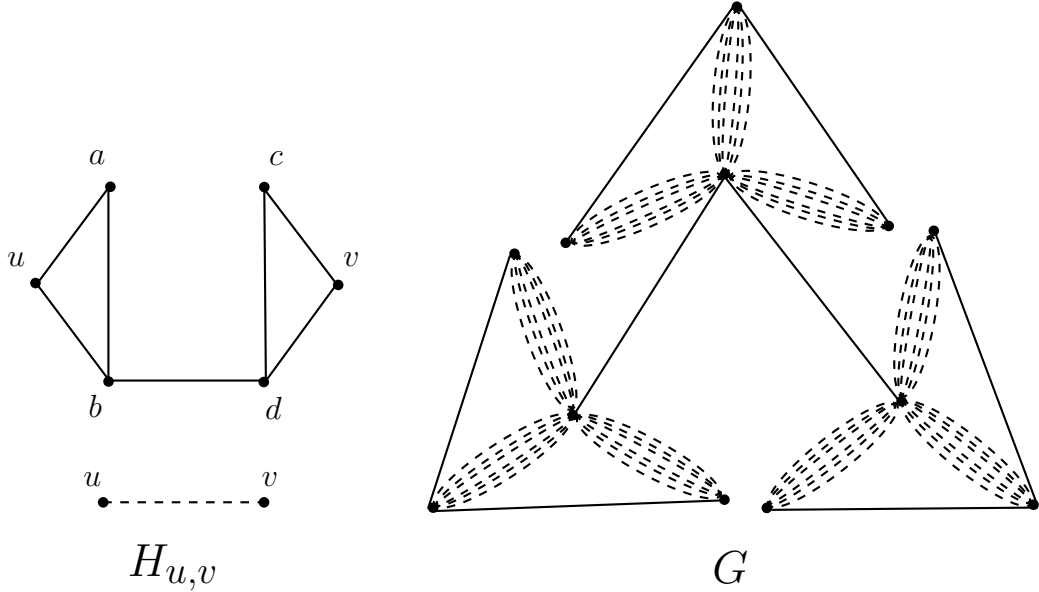


Figure 1.

A non- $(k, 1)$ -colorable planar graph  $G$  without 4-cycles and 5-cycles. The vertices  $a, b, c$ , and  $d$  cannot receive the same color 1. Now, we construct the graph  $S_z$  as follows. Let  $z$  be a vertex and  $x_1x_2x_3$  be a path. Take  $2k + 1$  copies  $H_{u_i, v_j}$  of  $H_{u,v}$  with  $1 \leq i \leq 2k + 1$  and  $1 \leq j \leq 3$ . Identify every  $u_i$  with  $z$  and identify  $v_j$  with  $x_j$ . Finally, we obtain  $G$  from three copies  $S_{z_1}, S_{z_2}$ , and  $S_{z_3}$  by adding the edges  $z_1z_2$  and  $z_2z_3$ . In every  $(1, k)$ -coloring of  $G$ , the path  $z_1z_2z_3$  contains a vertex  $z$  with color 2. In the copy of  $S_z$  corresponding to  $z$ , the path  $x_1x_2x_3$  contains a vertex  $x$  with color 2. Since each of  $z$  and  $x$  has at most  $k$  neighbors colored 2, one of  $2k + 1$  copies of  $H_{u,v}$  between  $z$  and  $x$ , does not contain a neighbor of  $z$  and  $x$  colored 2. This copy is not  $(1, k)$ -colorable, and thus  $G$  is not  $(1, k)$ -colorable.

## 4 Helpful Tools

Now, we investigate  $(d_1, d_2)$  such that  $G$  is  $(d_1, d_2)$ -colorable for every graph  $G$  without 4-cycles and 5-cycles. From two previous sections, we have that  $d_1, d_2 \geq 2$ . First, we present useful proposition and lemmas about a minimal planar graph  $G$  that is not  $(d_1, d_2)$ -colorable where  $d_1 \leq d_2$ .

**Proposition 1.** (a) *Each vertex  $v$  of  $G$  is a  $2^+$ -vertex.*

(b) *If  $v$  is a  $k$ -vertex has  $\alpha$  incident 3-faces,  $\beta$  adjacent good 2-vertices, and  $\gamma$  pendant 3-faces, then  $\alpha \leq \lfloor \frac{k}{2} \rfloor$  and  $2\beta + \alpha + \gamma \leq k$*

**Lemma 2.** [9] *Let  $G$  be  $(d_1, d_2)$ -colorable where  $d_1 \leq d_2$ .*

(a) *If  $v$  is a  $3^-$ -vertex, then at least two neighbors of  $v$  are  $(d_1 + 2)^+$ -vertices one of which is a  $(d_2 + 2)^+$ -vertex.*

(b) *If  $v$  is a  $(d_1 + d_2 + 1)^-$ -vertex, then at least one neighbor of  $v$  is a  $(d_1 + 2)^+$ -vertex.*

**Lemma 3.** *If a 2-vertex  $v$  is on a bad 3-face  $f$ , then the other face  $g$  which is incident to  $v$  is a  $7^+$ -face.*

**Proof.** Suppose that a face  $g$  is a  $6^-$ -face. Let a face  $f = uvw$ . By condition of  $G$ , a face  $g$  is neither 4, 5-face nor 3-face, otherwise  $G$  contains  $C_4$ . Now we suppose a face  $g$  is a 6-face and let  $g = u_1u_2u_3uvw$ . Since  $u$  is adjacent to  $w$ , there is a 5-cycle  $= u_1u_2u_3uw$ , a contradiction.  $\square$

**Lemma 4.** *Let  $f$  be a  $k$ -face where  $k \geq 7$ . Then,  $f$  has at most  $k - 6$  incident bad 2-vertices.*

**Proof.** By proof of Lemma 3, if a face  $f$  is incident to  $m$  bad 2-vertices, then there is a cycle  $C_{k-m}$  since we can add some edge to  $f$  to obtain a new cycle that has the length least than a face  $f$ .  $\square$

**Lemma 5.** *Let  $(u, v, w)$  be a bad 3-face  $f$  where  $d(u) = 2$ . Then at least one of following statements is true.*

(S1) *A vertex  $v$  is a  $(d_1 + 3)^+$ -vertex which has at least two  $(d_2 + 2)$ -neighbors.*

(S2) *A vertex  $w$  is a  $(d_2 + 3)^+$ -vertex which has at least two  $(d_1 + 2)$ -neighbors.*

(S3) *A vertex  $v$  or a vertex  $w$  is a  $(d_1 + d_2 + 2)^+$ -vertex.*

**Proof.** Assume  $c$  is a  $(d_1, d_2)$ -coloring in  $G - u$ . If two neighbors of  $u$  share the same color, then we can color  $u$  by  $\{1, 2\} - \{c(v)\}$ . So  $c(v) \neq c(w)$ . By symmetry let  $c(v) = 1$  and

$c(w) = 2$ . By Lemma 2, we have a vertex  $v$  is a  $(d_1 + 2)^+$  and a vertex  $w$  is a  $(d_2 + 2)^+$ . Then  $v$  has  $d_1$  neighbors of color 1 to forbid  $u$  from being colored by 1 and  $w$  has  $d_2$  neighbors of color 2 to forbid  $u$  from being colored by 2. Next, to avoid recoloring  $v$  by 2 and  $w$  by 1. Then  $v$  has one neighbor with color 2 which has  $d_2$  neighbors of color 2 or  $v$  has  $d_2$  neighbors with color 2. Otherwise,  $w$  has one neighbor with color 1 which has  $d_1$  neighbors of color 1 or  $w$  has  $d_1$  neighbors with color 1.  $\square$

## 5 $(4, 4)$ -coloring

**Theorem 3.** *If  $G$  is a planar graph without cycles of length 4 or 5, then  $G$  is  $(4, 4)$ -colorable.*

**Proof.** Suppose that  $G$  is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex  $u$  in  $G$  be  $\mu(u) = 2d(u) - 6$  and the initial charge of a face  $f$  in  $G$  be  $\mu(f) = d(f) - 6$ . Then by Euler's formula  $|V(G)| - |E(G)| + |F(G)| = 2$  and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$

Now, we establish a new charge  $\mu^*(x)$  for all  $x \in V(G) \cup F(G)$  by transferring charge from one element to another and the summation of new charge  $\mu^*(x)$  remains  $-12$ . If the final charge  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ , then we get a contradiction and the prove is completed.

The discharging rules are

- (R1) Every  $6^+$ -vertex sends charge 1 to each adjacent good 2-vertex.
- (R2) Every  $6^+$ -vertex sends charge 2 to each incident 3-face.
- (R3) Every  $6^+$ -vertex sends charge 1 to each adjacent pendant 3-face.
- (R4) Every  $7^+$ -face sends charge 1 to each incident bad 2-vertex.
- (R5) Every 4-vertex or 5-vertex sends charge 1 to each incident 3-face.
- (R6) Every bad 3-face sends charge 1 to each incident 2-vertex.

It remains to show that resulting  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ .

It is evident that  $\mu^*(x) = \mu(x) = 0$  if  $x$  is a 3-vertex or a 6-face.

Now, let  $v$  be a  $k$ -vertex.

For  $k = 2$ , a vertex  $v$  has two  $6^+$ -neighbors by Lemma 2. If  $v$  is a good 2-vertex, then  $\mu^*(v) \geq \mu(v) + 2 \cdot 1 = 0$  by (R1). If  $v$  is a bad 2-vertex, then  $v$  is incident to a  $7^+$ -face by Lemma 3. Thus  $\mu^*(v) \geq \mu(v) + 1 + 1 = 0$  by (R4) and (R6).

For  $k = 4, 5$ , by Proposition 1 (b), a vertex  $v$  is incident to at most two 3-faces. By (R5),  $\mu^*(v) \geq \mu(v) - 2 \cdot 1 \geq 0$ .

Consider  $k = 6^+$ . Let  $v$  have  $\alpha$  incident 3-faces,  $\beta$  adjacent good 2-vertices, and  $\gamma$  pendant 3-faces. By Proposition 1 (b), we have  $2\alpha + \beta + \gamma \leq d(v)$ . Moreover,  $\mu(v) = 2d(v) - 6 \geq d(v)$  if  $d(v) \geq 6$ . Thus, by (R1), (R2), and (R3), we have  $\mu^*(v) = \mu(v) - (2\alpha + \beta + \gamma) \geq 0$ .

Now let  $f$  be a  $k$ -face.

For  $k = 7^+$ , by Lemma 4, a  $k$ -face  $f$  has at most  $k - 6$  incident bad 2-vertices. By (R4),  $\mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0$ .

Consider  $k = 3$ . If  $f$  is a bad 3-face, then we have  $f = (2, 6^+, 6^+)$ -face by Lemma 2. Then by (R2) and (R6),  $\mu^*(f) \geq \mu(f) + 2 \cdot 2 - 1 = 0$ . Now, It remains to consider a good 3-face. If  $f$  is incident to a  $4^+$ -vertex and a  $6^+$ -vertex, then  $\mu^*(f) \geq \mu(f) + 2 + 1 \geq 0$  by (R2) and (R5). If  $f$  is a  $(3, 3, 6^+)$ -face, then the pendant neighbor of a 3-vertex is a  $6^+$ -vertex by Lemma 2. Thus  $\mu^*(f) \geq \mu(f) + 2 + 1 + 1 \leq 0$  by (R2) and (R3). Finally, if  $f$  is a  $(4^+, 4^+, 4^+)$ -face, then  $\mu^*(f) \geq \mu(f) + 3 \cdot 1 \leq 0$  by (R5).

Since  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ , this completes the proof.  $\square$

## 6 $(3, 5)$ -coloring

**Theorem 4.** *If  $G$  is a planar graph without cycles of length 4 or 5, then  $G$  is  $(3, 5)$ -colorable.*

**Proof.** Suppose that  $G$  is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex  $u$  in  $G$  be  $\mu(u) = 2d(u) - 6$  and the initial charge of a face  $f$  in  $G$  be  $\mu(f) = d(f) - 6$ . Then by Euler's formula  $|V(G)| - |E(G)| + F(G) = 2$  and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$

Now, we establish a new charge  $\mu^*(x)$  for all  $x \in V(G) \cup F(G)$  by transferring charge from one element to another and the summation of new charge  $\mu^*(x)$  remains  $-12$ . If the final charge  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ , then we get a contradiction and the prove is completed.

The discharging rules are

- (R1) Every 5-vertex sends charge  $\frac{4}{5}$  to each adjacent good 2-vertex.
- (R2) Every 5-vertex sends charge  $\frac{8}{5}$  to each incident 3-face.
- (R3) Every 5-vertex sends charge  $\frac{4}{5}$  to each adjacent pendant 3-face.

- (R4) Every 6-vertex sends charge 1 to each adjacent good 2-vertex.
- (R5) Every 6-vertex or 7-vertex sends charge 2 to each incident 3-face.
- (R6) Every 6-vertex sends charge 1 to each adjacent pendant 3-face.
- (R7) Every  $7^+$ -vertex sends charge  $\frac{6}{5}$  to each adjacent good 2-vertex.
- (R8) Every  $8^+$ -vertex sends charge  $\frac{12}{5}$  to each incident 3-face.
- (R9) Every  $7^+$ -vertex sends charge  $\frac{6}{5}$  to each adjacent pendant 3-face.
- (R10) Every  $7^+$ -face sends charge 1 to each incident bad 2-vertex.
- (R11) Every 4-vertex sends charge 1 to each incident 3-face.
- (R12) Every bad 3-face sends charge 1 to each incident 2-vertex.

Next, we show that the final charge  $\mu^*(u)$  is nonnegative.

It is evident that  $\mu^*(x) = \mu(x) = 0$  if  $x$  is a 3-vertex or a 6-face.

Now, let  $v$  be a  $k$ -vertex.

For  $k = 2$ , a vertex  $v$  has two  $5^+$ -neighbors one of which is a  $7^+$ -neighbor by Lemma 2. If  $v$  is a good 2-vertex, then  $\mu^*(v) \geq \mu(v) + \frac{4}{5} + \frac{6}{5} = 0$  by (R1) and (R7). If  $v$  is a bad 2-vertex, then  $v$  is incident to a  $7^+$ -face by Lemma 3. Thus  $\mu^*(v) \geq \mu(v) + 1 + 1 = 0$  by (R10) and (R12).

For  $k = 4$ , by Proposition 1 (b), a vertex  $v$  is incident to at most two 3-faces. By (R11),  $\mu^*(v) \geq \mu(v) - 2 \cdot 1 \geq 0$ .

Consider  $k = 5$ . Let  $v$  have  $\alpha$  incident 3-faces,  $\beta$  adjacent good 2-vertices, and  $\gamma$  pendant 3-faces. By Proposition 1 (b),  $2\alpha + \beta + \gamma \leq d(v)$ . Moreover, we have  $\frac{8}{5}\alpha + \frac{4}{5}\beta + \frac{4}{5}\gamma = \frac{4}{5}(2\alpha + \beta + \gamma) \leq \frac{4}{5}d(v)$  and  $\mu(v) = 2d(v) - 6 = \frac{4}{5}d(v)$  if  $d(v) = 5$ . Thus by (R1), (R2), and (R3), we have  $\mu^*(v) = \mu(v) - (\frac{8}{5}\alpha + \frac{4}{5}\beta + \frac{4}{5}\gamma) \geq 0$ .

Consider  $k = 6$ . Let  $v$  have  $\alpha$  incident 3-faces,  $\beta$  adjacent good 2-vertices, and  $\gamma$  pendant 3-faces. By Proposition 1 (b), we have  $2\alpha + \beta + \gamma \leq d(v)$ . Moreover,  $\mu(v) = 2d(v) - 6 = d(v)$  if  $d(v) = 6$ . Thus, by (R4), (R5), and (R6), we have  $\mu^*(v) = \mu(v) - (2\alpha + \beta + \gamma) = 0$ .

Consider  $k = 7$ . If  $v$  is not incident to a 3-face, then we have  $\mu^*(v) = \mu(v) - 6 \cdot \frac{6}{5} \geq 0$  by Lemma 2, (R7), and (R9). If  $v$  is incident to one 3-face, then we have  $\mu^*(v) = \mu(v) - (2 + 5 \cdot \frac{6}{5}) = 0$  by (R5), (R7), and (R9). If  $v$  is incident to two 3-faces, then we have  $\mu^*(v) = \mu(v) - (2 \cdot 2 + 3 \cdot \frac{6}{5}) \geq 0$  by (R5), (R7), and (R9). Finally, if  $v$  is incident to three 3-faces, then we have  $\mu^*(v) = \mu(v) - (3 \cdot 2 + \frac{6}{5}) \geq 0$  by (R5), (R7) and (R9).

Consider  $k = 8^+$ . Let  $v$  have  $\alpha$  incident 3-faces,  $\beta$  adjacent good 2-vertices, and  $\gamma$  pendant 3-faces. By Proposition 1 (b),  $2\alpha + \beta + \gamma \leq d(v)$ . Moreover, we have  $\frac{12}{5}\alpha + \frac{6}{5}\beta + \frac{6}{5}\gamma =$



$\frac{6}{5}(2\alpha + \beta + \gamma) \leq \frac{6}{5}d(v)$  and  $\mu(v) = 2d(v) - 6 \geq \frac{6}{5}d(v)$  if  $d(v) \geq 8$ . Thus by (R7), (R8), and (R9), we have  $\mu^*(v) = \mu(v) - (\frac{12}{5}\alpha + \frac{6}{5}\beta + \frac{6}{5}\gamma) \geq 0$ .

Now let  $f$  be a  $k$ -face.

For,  $k = 7^+$ . By Lemma 4, a  $k$ -face  $f$  has at most  $k - 6$  incident bad 2-vertices. By (R11),  $\mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0$ .

Consider  $k = 3$ . If  $f$  is a bad 3-face, then we have  $f$  is a  $(2, 6^+, 6^+)$ -face or  $f$  is a  $(2, 5^+, 8^+)$  by Lemma 5. Then by (R2), (R5), (R8), and (R12),  $\mu^*(f) \geq \mu(f) + 2 \cdot 2 - 1 = 0$  or  $\mu^*(f) \geq \mu(f) + \frac{8}{5} + \frac{12}{5} - 1 = 0$ . Now, it remains to consider a good 3-face. If  $f$  is incident to a  $4^+$ -vertex and a  $6^+$ -vertex, then  $\mu^*(f) \geq \mu(f) + 2 + 1 \geq 0$  by (R5) and (R11). If  $f$  is a  $(3, 3, 7^+)$ -face, then the pendant neighbor of a 3-vertex is a  $5^+$ -vertex by Lemma 2. Thus  $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{4}{5} + 2 \geq 0$  by (R3) and (R5). If  $f$  is a  $(3, 3, 5^+)$ -face, then the pendant neighbor of a 3-vertex is a  $7^+$ -vertex by Lemma 2. Thus  $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{6}{5} + \frac{8}{5} \geq 0$  by (R2) and (R7). Finally, if  $f$  is a  $(4^+, 4^+, 4^+)$ -face, then  $\mu^*(f) \geq \mu(f) + 3 \cdot 1 \geq 0$  by (R11).

Since  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ , this completes the proof.  $\square$

## 7 $(2, 9)$ -coloring

**Theorem 5.** *If  $G$  is a planar graph without cycles of length 4 or 5, then  $G$  is  $(2, 9)$ -colorable.*

**Proof.** Suppose that  $G$  is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex  $u$  in  $G$  be  $\mu(u) = 2d(u) - 6$  and the initial charge of a face  $f$  in  $G$  be  $\mu(f) = d(f) - 6$ . Then by Euler's formula  $|V(G)| - |E(G)| + F(G) = 2$  and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$

Now, we establish a new charge  $\mu^*(x)$  for all  $x \in V(G) \cup F(G)$  by transferring charge from one element to another and the summation of new charge  $\mu^*(x)$  remains  $-12$ . If the final charge  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ , then we get a contradiction and the prove is completed.

The discharging rules are

- (R1) Every  $k$ -vertex for  $4 \leq k \leq 10$  sends charge  $\frac{1}{2}$  to each adjacent good 2-vertex.
- (R2) Every 4-vertex sends charge 1 to each incident 3-face.
- (R3) Every  $k$ -vertex for  $4 \leq k \leq 10$  sends  $\frac{1}{2}$  to each adjacent pendant 3-face.

- (R4) Every  $k$ -vertex for  $5 \leq k \leq 10$  sends charge  $\frac{3}{2}$  to each incident 3-face.
- (R5) Every 11-vertex sends charge  $\frac{5}{2}$  to each incident 3-face.
- (R6) Every  $11^+$ -vertex sends charge  $\frac{3}{2}$  to each adjacent good 2-vertex.
- (R7) Every  $12^+$ -vertex sends charge 3 to each incident 3-face.
- (R8) Every  $11^+$ -vertex sends charge  $\frac{3}{2}$  to each adjacent pendant 3-face.
- (R9) Every  $7^+$ -face sends charge 1 to each incident bad 2-vertex.
- (R10) Every bad 3-face sends charge 1 to each incident 2-vertex.

Next, we show that the final charge  $\mu^*(u)$  is nonnegative.

It is evident that  $\mu^*(x) = \mu(x) = 0$  if  $x$  is a 3-vertex or a 6-face.

Now, let  $v$  be a  $k$ -vertex.

For  $k = 2$ , a vertex  $v$  has two  $4^+$ -neighbors one of which is a  $11^+$ -neighbor by Lemma 2. If  $v$  is a good 2-vertex, then  $\mu^*(v) \geq \mu(v) + \frac{1}{2} + \frac{3}{2} = 0$  by (R1) and (R6). If  $v$  is a bad 2-vertex, then  $v$  is incident to a  $7^+$ -face by Lemma 3. Thus  $\mu^*(v) \geq \mu(v) + 1 + 1 = 0$  by (R9) and (R10).

Consider  $k = 4$ . Let  $v$  have  $\alpha$  incident 3-faces,  $\beta$  adjacent good 2-vertices, and  $\gamma$  pendant 3-faces. By Proposition 1 (b),  $2\alpha + \beta + \gamma \leq d(v)$ . Moreover, we have  $\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = \frac{1}{2}(2\alpha + \beta + \gamma) \leq \frac{1}{2}d(v)$  and  $\mu(v) = 2d(v) - 6 = \frac{1}{2}d(v)$  if  $d(v) = 4$ . Thus by (R1), (R2), and (R3), we have  $\mu^*(v) = \mu(v) - (\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma) \geq 0$ .

Consider  $k$  for  $5 \leq k \leq 10$ . By (R1), (R3), and (R4), we show only the case that  $v$  has  $\lfloor \frac{d(v)}{2} \rfloor$  incident 3-faces because this case has final charge less than the other cases. Consider  $\frac{3}{2} \frac{d(v)}{2} \leq 2d(v) - 6$ , then we have  $d(v) \geq 5$  because two times charge in (R1) or (R3) is less than charge in (R4). Thus we have  $\mu^*(v) \geq 0$ .

Consider  $k = 11$ . By (R5), (R6), and (R8), we show only the case that  $v$  is not incident to 3-face because this case has final charge less than the other cases. we have  $\mu^*(v) = 16 - 10(\frac{3}{2}) \geq 0$ . If there is one 3-face, then  $\mu^*(v) = 16 - (9(\frac{3}{2}) + \frac{5}{2}) = 0$ .

Now let  $f$  be a  $k$ -face.

For  $k = 7^+$ . By Lemma 4, a  $k$ -face  $f$  has at most  $k - 6$  incident bad 2-vertices. By (R9),  $\mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0$ .

Consider  $k = 3$ . If  $f$  is a bad 3-face, then we have  $f$  is a  $(2, 4^+, 12^+)$ -face or  $f$  is a  $(2, 5^+, 11^+)$  by Lemma 5. Then by (R2), (R4), (R5), and (R7),  $\mu^*(f) \geq \mu(f) + 1 + 3 - 1 = 0$  or  $\mu^*(f) \geq \mu(f) + \frac{3}{2} + \frac{5}{2} - 1 = 0$ . Now, it remains to consider a good 3-face. Consider  $f$  is incident to exactly one 3-vertex. If  $f$  is not incident to a  $11^+$ -vertex, then pendant neighbor of a 3-vertex is a  $11^+$ -vertex by Lemma 2. Thus  $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{1}{2} + \frac{3}{2} \geq 0$  by (R2) and (R8). If  $f$  is incident to a  $4^+$ -vertex and a  $11^+$ -vertex, then  $\mu^*(f) \geq \mu(f) + \frac{1}{2} + \frac{5}{2} \geq 0$

by (R2) and (R5). If  $f$  is a  $(3, 3, 11^+)$ -face, then the pendant neighbor of a 3-vertex is a  $4^+$ -vertex by Lemma 2. Thus  $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{1}{2} + \frac{5}{2} \geq 0$  by (R3) and (R5). If  $f$  is a  $(3, 3, 4^+)$ -face, then the pendant neighbor of a 3-vertex is a  $11^+$ -vertex by Lemma 2. Thus  $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{3}{2} + 1 \geq 0$  by (R2) and (R8). Finally, if  $f$  is a  $(4^+, 4^+, 4^+)$ -face, then  $\mu^*(f) \geq \mu(f) + 3 \cdot 1 \geq 0$  by (R2).

Since  $\mu^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ , this completes the proof.  $\square$

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